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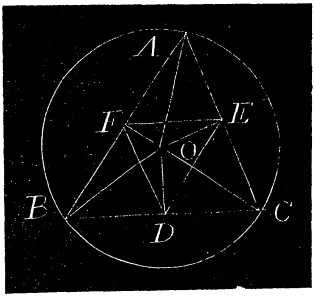
Proposition.—"$D+d = a\cot A + b\cot B + c\cot C$, in which d and D are the diameters of the circles inscribed in, and circumscribed about, a triangle whose sides are a, b, c, and opposite angles A, B, C."

Demonstration.—Let ABC represent the triangle and O, the center of the circumscribing circle, and let D, E, F, represent the middle points of the sides a, b, c.



Because the $\angle BOC$ at the center is double the $\angle A$ at the circumference, $\angle BOD = \angle A$.

$$\therefore \quad DO\ (=x) = \tfrac{1}{2}a\cot A, \text{ and similarly}$$
$$EO\ (=y) = \tfrac{1}{2}b\cot B,$$
$$FO\ (=z) = \tfrac{1}{2}c\cot C.$$

Join DE, DF and EF; then is $EF = \frac{1}{2}a$, $DF = \frac{1}{2}b$ and $DE = \frac{1}{2}c$.

Because the quadrilaterals $DCEO$, $EAFO$ and $FBDO$ have, each, the sum of their opposite angles $= 180°$ they are inscribable in a circle:

Therefore (Eucl. D. VI.) we have

$$\tfrac{1}{2}aR = \tfrac{1}{2}cy + \tfrac{1}{2}bz,$$
$$\tfrac{1}{2}bR = \tfrac{1}{2}cx + \tfrac{1}{2}az,$$
$$\tfrac{1}{2}cR = \tfrac{1}{2}ay + \tfrac{1}{2}bx.$$
$$\therefore \quad R(a+b+c) = (b+c)x + (a+c)y + (a+b)z, \tag{1}$$

where R is the radius of the circumscribing circle.

Let r represent the radius of the inscribed circle, then, obviously,

$$r(a+b+c) = ax + by + cz. \tag{2}$$

Adding (1) and (2) we get

$$(a+b+c)(R+r) = (a+b+c)x + (a+b+c)y + (a+b+c)z,$$

or

$$R + r = x + y + z = \tfrac{1}{2}a\cot A + \tfrac{1}{2}b\cot B + \tfrac{1}{2}c\cot C.$$
$$\therefore \quad D+d = a\cot A + b\cot B + c\cot C.$$

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received as follows:

From Marcus Baker, 162 and 164; Prof. L. G. Barbour, 162; Prof. W. P. Casey, 166; George Eastwood, 164 and 166; Prof. H. T. Eddy, 165 & 168; Henry Gunder, 162; W. E. Heal, 162; Chas. H. Kummell, 162, 168 and 169; Prof. O. Pratt, Sen., 163; J. W. P. Reid, 162; L. Regan, 162; E. B. Seitz, 162 and 164; Prof. C. M. Woodward, 167; A. W. Whitaker, 162; F. A. Walker, 162.

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162. "Given

$$x^2 + xy + y^2 = 37, \ (1)$$
$$x^2 + xz + z^2 = 49, \ (2)$$
$$y^2 + yz + z^2 = 61, \ (3)$$

to find x, y and z by quadratics".

SOLUTION BY W. E. HEAL, WHEELING, INDIANA.

From (1), (2) and (3) we get, by multiplication,

$$x^3 - y^3 = 37(x - y), \tag{4}$$
$$x^3 - z^3 = 49(x - z), \tag{5}$$
$$y^3 - z^3 = 61(y - z). \tag{6}$$

$\therefore$ $$37(x - y) = 49(x - z) - 61(y - z). \tag{7}$$

Reducing, $$2y = x + z. \tag{8}$$

Put $x = y - v$ (9), and $z = y + v$. (10)

Substituting in (3) and (2) we have

$$3y^2 + 3yv + v^2 = 61, \tag{11}$$
$$3y^2 \qquad + v^2 = 49. \tag{12}$$

$\therefore$ $$3yv = 12, \text{ or, } yv = 4. \tag{13}$$

From (12) and (13) $\sqrt{[(12)^2 - 12(13)^2]} = \pm(3y^2 - v^2) = 47;$ (14)

" (12) " (14) $\sqrt{[\frac{1}{6}(12) + \frac{1}{6}(14)]} = y = \pm 4 \text{ or } v = \pm 4\sqrt{3},$ (15)

$\sqrt{[\frac{1}{6}(12) - \frac{1}{6}(14)]} = v = \pm 1 \text{ or } y = \pm \frac{1}{3}\sqrt{3}.$ (16)

$\therefore$ $x = \pm 3 \text{ or } \mp \frac{11}{3}\sqrt{3}, \text{ and } z = \pm 5 \text{ or } \pm \frac{13}{3}\sqrt{3}.$

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163 "Resolve the first member of the general Cubic Equation, $x^3 + px^2 + qx = -r$, into three factors, such that, when their signs are changed, their sum shall equal p; when their signs are unchanged the sum of their products, taken two and two, shall equal q; and when their signs are unchanged, their continued product shall $= -r$. Or, in other words, find the forms of the three roots in terms of x and the coefficients."

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

The following three factors fulfil all the conditions required in the problem, and, therefore, must be the true forms of the three roots:

$$x;$$
$$-\tfrac{1}{2}p - \tfrac{1}{2}x + \tfrac{1}{2}\sqrt{[p^2 - 4q - 2px - 3x^2]};$$
$$-\tfrac{1}{2}p - \tfrac{1}{2}x - \tfrac{1}{2}\sqrt{[p^2 - 4q - 2px - 3x^2]}.$$

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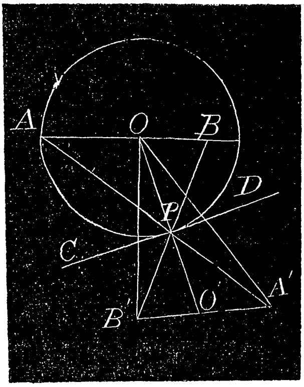
164. "From any two points to draw two lines which shall meet in the circumference of a given circle, and make equal angles with the tangent at the point of intersection."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Analysis: Suppose the problem solved; A and B the given points, O in AB the centre of the given circle, P the required point in its circumference and AP, BP lines drawn making equal angles APC, BPD with the indefinite tangent CD.



Join OP; then OP is perpendicular to CD (Eucl. III. 18), the angles OPC, OPD are right angles, the angles APC, BPD are equal (by hypothesis), therefore $\angle APO = \angle BPO$. Hence PO divides AB in O so that

$$AO : OB :: AP : BP. \text{ (Eucl. VI. 3.)}$$

Now AO and OB are given lines, therefore $AP : PB$ is a given ratio. Let it be as $1 : n$, so that $n.AP = BP$. But (Eucl. VI. B.)

$$AP.PB = AO.OB + OP^2,$$
$$= AO(AO+BO) = AO.AB,$$

a given magnitude, therefore $AP^2 = AO \times AB \div n$, a given rectangle, therefore AP is known. Hence the following

Construction. On AB, the line joining the given points, take AO equal to the radius of the given circle. With centre A and a radius equal to a mean proportional between AO and the nth part of AB, describe an arc intersecting the given circumference in P. Join OP and erect the indefinite perpendicular CPD. Join AP, BP, then the angles they respectively make with CPD will be equal to each other.

Scholium. Let AP be produced (downward) to A', BP to B', and OP to O' in $A'B'$. Then if A' and B' are taken for the given points, it may be easily shown that the lines $A'P$ and $B'P$ will make equal angles with the tangent CPD.

[Because, in the foregoing solution, one of the given points must lie in the circumference of the given circle, therefore, if by "a given circle" is meant a circle given *both in magnitude and position*, this solution is but a particular case of a more general solution involving an equation of the 4th degree. This is the view taken by E. B. Seitz and Marcus Baker, each of whom has reduced the solution to an algebraic equation of the 4th degree. Mr. Baker remarks, "This is the same as the celebrated problem of Alhazen. It was the prize question in the Ladies' Diary for 1727. Solutions of it have been given by Slusius, Huyghens, Dr. Robert Smith, Robert Simson, Robins, Hutton, Wales, &c."—Ed.]

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165. "Of the system of dynamical equations,

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=0,\quad \frac{d^2y}{dt^2}+\frac{my}{r^3}=0,\quad \frac{d^2z}{dt^2}+\frac{mz}{r^3}=0,$$

where $r=\surd(x^2+y^2+z^2)$, seven first integrals are obtained of which it is subsequently found that five only are independent. How many final integrals can hence be deduced without proceeding to another integration."

SOLUTION BY PROF. H. T. EDDY, UNIVERSITY OF CINCINNATI.

The differential equations of the relative motion of two particles, having a combined mass of m, due to their mutual gravitation, are:

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=0,\quad \frac{d^2y}{dt^2}+\frac{my}{r^3}=0,\quad \frac{d^2z}{dt^2}+\frac{mz}{r^3}=0,$$

in which $r^2=x^2+y^2+z^2$. Multiply severally by dx, dy, dz, and iategrate

$$\therefore\qquad \frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}=\frac{ds^2}{dt^2}=\frac{2m}{r}+a.\qquad (1)$$

Again, multiply severally by either x, y or z and substitute, i. e., eliminate r,

$$\therefore\qquad z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}=0,\ \&c.$$

$$\therefore\quad z\frac{dy}{dt}-y\frac{dz}{dt}=b_1\ (2),\quad x\frac{dz}{dt}-z\frac{dx}{dt}=b_2\ (3),\quad y\frac{dx}{dt}-x\frac{dy}{dt}=b_3.\ (4)$$

Once more, multiply the given equations severally by either (2), (3) or (4) and substitute,

$$\therefore\quad b_2\frac{d^2z}{dt^2}-b_3\frac{d^2y}{dt^2}=-\left(x\frac{dz}{dt}-z\frac{dx}{dt}\right)\frac{mz}{r^3}+\left(y\frac{dx}{dt}-x\frac{dy}{dt}\right)\frac{my}{r^3}$$

$$\therefore\quad b_2\frac{dz}{dt}-b_3\frac{dy}{dt}=m\int\left[\frac{x^2+y^2+z^2}{r^2}dx-x(xdx+ydy+zdz)\right]$$

$$\therefore\quad b_2\frac{dz}{dt}-b_3\frac{dy}{dt}=m\int\frac{rdx-xdr}{r^2}=m\int d\frac{x}{r}.$$

$$\therefore\qquad b_2\frac{dz}{dt}-b_3\frac{dy}{dt}=\frac{mx}{r}+c_1,\qquad (5)$$

$$b_3\frac{dx}{dt}-b_1\frac{dz}{dt}=\frac{my}{r}+c_2,\qquad (6)$$

$$b_1\frac{dy}{dt}-b_2\frac{dx}{dt}=\frac{mz}{r}+c_3.\qquad (7)$$

Equations (1) to (7) inclusive are the seven first integrals.

Now, multiply severally (2), (3), (4), by x, y, z and add,

$$\therefore\qquad b_1x+b_2y+b_3z=0.\qquad (8)$$

Also, multiply (5), (6), (7) severally by x, y, z, add and reduce by (2), (3), (4),

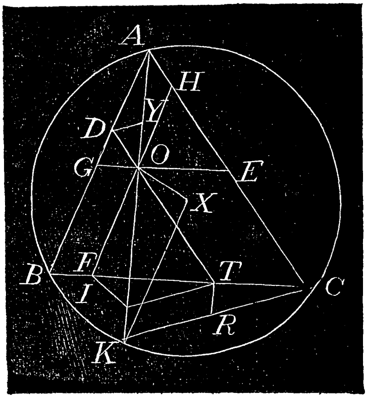
$$\therefore\qquad mr+c_1x+c_2y+c_3z=b_1^2+b_2^2+b_3^2.\qquad (9)$$

Equations (8) and (9) represent a plane and quadric respectively whose intersection is the orbit: there are therefore no other independent relations between x, y, z. There must be however three more equations expressing xyz in terms of t: these cannot however be determined by any elimination between the seven first integrals, because t only appears in them as a differential. Only two of these five independent primitives can therefore be obtained without further integration.

166. "If through a point O, within a given triangle ABC, three lines be drawn respectively parallel to the sides of the triangle; viz., GE parallel to BC, FH, to AB and DT to AC, and there is given, $GO\times OE+FO\times OH+DO\times OT$; to find the locus of O by plane geometry."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Construction. Let X be the centre of the circle circumscribing the triangle ABC. Join AO and produce it to K, draw TR parallel to OK, and TI parallel to CK. Join IF; and make the angle $ADY=$ angle OIF.



Demonstration. It is evident that a circle will circumscribe the quadrilateral OF IT, and that the triangles GOA and TRC are similar; also, the triangles ODY and OIT are similar. Hence we have $GO\times TC=AO\times TR$, or $GO\times OE=AO\times IK$ (1); and from the similar triangles ADY and OIF we have $AD\times OF=AY\times OI$, and therefore $HO\times OF=AY\times OI$ (2); and from the similar triangles ODY and OIT we have $DO\times OT=OY\times OI$ (3). By adding equations (1), (2) and (3) we get $GO\times OE+HO\times OF+DO\times OT=AO\times IK+AY\times OI+OY\times OI=AO\times IK+AO\times OI=AO\times OK=(XK)^2-(XO)^2=$ a given quantity.

[Solutions of the remaining problems in No. 3 will be published in No. 5, as follows: 167, by Prof. C. M. Woodward; 168, by Prof. H. T. Eddy; 169, by Chas. H. Kummell.–No answers to Queries in No. 3 have been rec'd.]

PROBLEMS.

160. (Selected) By G. W. Hill. — Given the lengths of the eight edges of a quadrangular pyramid to find its altitude.

161. By O. H. Merrill, South Rutland, N. Y. — A pail in the form of a frustrum of a cone—radius of upper base $2r$, of lower base r, and height h—is inclined so that if water be poured in to it, the water will just reach the lower edge of the upper base, and the upper edge of the lower base. How many gallons of water in the pail?

162. By Artemas Martin, Erie, Pa. — Find the least integral values of x and y that will satisfy the equation $x^2-9817y^2=1$.